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Loose vertices in C_4-free Berge graphs

I. Parfenoff, F. Roussel, I. Rusu*

Université d'Orléans, L.I.F.O., B.P. 6759, 45067 Orléans Cedex 2, France

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Abstract

Following Maire (Graphs Combin. 10 (3) (1994) 263) we call a *loose vertex* a vertex whose neighbourhood induces a P_4-free graph, and we show that every C_4-free Berge graph G which is not a clique either is breakable (i.e. G or $\bar{G}$ has a star-cutset) or contains at least two non-adjacent loose vertices. Consequently, every minimal imperfect C_4-free graph has loose vertices.

Keywords: Lexicographic breadth-first search; Perfect graph; Berge graph

1. Introduction

An important number of results have been obtained concerning *perfect graphs*, i.e. graphs in which every induced subgraph has the clique number equal to the chromatic number. A large part of these results is devoted to particular cases of the Strong Perfect Graph Conjecture (SPGC), which claims that a graph is perfect if and only if it is a *Berge graph*, that is, it contains no C_{2k+1} and no $\overline{C_{2k+1}}$, $k \geqslant 2$ (where C_p is the classical notation for a chordless cycle with p vertices). These results bring more and more information on perfect graphs, but they cannot provide, until the date and despite numerous attempts, a proof of the SPGC for C_4-free graphs. In other words, it is not known whether C_4-free Berge graphs are perfect or not. [1]

* Corresponding author. I.R.I.N., Universite de Nantes, 2 rue de la Houssiniere, BP 92208, 44322, Nantes Cedex 3, France.

E-mail address: rusu@irin.univ-nantes.fr (I. Rusu).

[1] In fact, after this paper was accepted for publication, the authors learned that Conforti, Cornuéjols and Vušković proved the perfection of C_4-free Berge graphs, in a 70-pages paper which is submitted for publication. We think that the result we present here is still important since it presents a new property of C_4-free Berge graphs, which could help either to give a recognition algorithm for these graphs (such an algorithm is not known) or to give an alternative proof of their perfection.

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Often, this kind of results is approached using *minimal imperfect graphs*: these are graphs which are not perfect but whose proper induced subgraphs are. Then, it is assumed that the class which has to be proved perfect contains a minimal imperfect graph G and it is shown that G has a property that minimal imperfect graphs cannot have (it has a star-cutset [3], an even pair [9], a special vertex [14] etc.), and a contradiction is obtained.

One of these properties interests us in this paper. Define a *star-cutset* of a graph $G=(V,E)$ to be a set S of vertices such that the graph induced by $V-S$ is disconnected, and there exists a vertex $s\in S$ adjacent in G to all the other vertices in S. Chvátal [3] showed that:

Star-cutset Lemma. *No minimal imperfect graph has a star-cutset.*

Since a graph is minimal imperfect if and only if its complement is [7], to show that a graph G is not minimal imperfect it is sufficient to show that either G or its complement $\bar{G}$ has a star-cutset (we then say that G is *breakable*; in the contrary case it is *unbreakable*).

In this paper, we show that C_4-free Berge graphs either are breakable, or they have a *loose vertex* (a vertex whose neighbourhood induces no chordless path on four vertices, denoted P_4). By the Star-cutset Lemma, a consequence of this result is that a minimal imperfect C_4-free graph G always has a loose vertex (for C_{2k+1} this is obviously true; $\overline{C_{2k+1}}$, $k\geqslant 2$, is not C_4-free). Moreover, as a P_4-free graph is disconnected or has disconnected complement [13], and since the last possibility implies, in our C_4-free graph, the existence of a star-cutset in $\bar{G}$, we deduce that a minimal imperfect C_4-free graph always has a vertex whose neighbourhood is non-connected and P_4-free. This is not sufficient yet to deduce the SPGC for C_4-free graphs, but the step we make is important, since the results we have at the date on C_4-free Berge graphs are *very* poor.

The paper is organized as follows. In Section 2, we give details on the algorithm LexBFS and the properties of the ordering obtained by applying LexBFS on an arbitrary graph. In Section 3, the main theorem is proved, using claims whose proofs are given in Section 4. Section 5 contains the conclusion.

2. Graph decomposition using LexBFS

The algorithm LexBFS has been introduced in [11], and produces an ordering of the vertices such that the last vertex in this order is often a special vertex (see [11], [2], [12]). Our aim is to show that for C_4-free Berge graphs the last vertex in the order is a loose vertex, or else G is breakable.

To describe the algorithm, we let every vertex have a label which is initially empty, and is modified during the algorithm by adding integers in decreasing (from left to right) order. The labels are compared using lexicographic order. Here is the

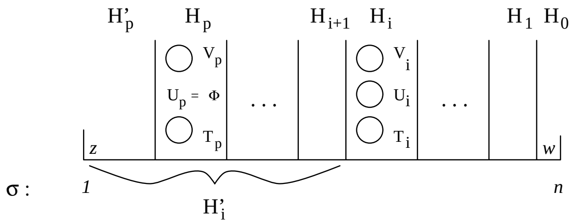

Fig. 1. The sets H_i, H_i' $(i=0,1,\ldots,p)$.

algorithm:

Algorithm LexBFS

Input: a connected graph $G=(V,E)$ with n vertices
Output: a function $\sigma:\{1,2,\ldots,n\}\to V$ (an order on V)

assign the label $\emptyset$ to each vertex;
for $i:=n$ down to 1 do
pick an unnumbered vertex v with a largest label (in lexicographic order);
$\sigma(i):=v$; {comment: this assigns to v the number i}
for each unnumbered vertex $t\in N(v)$ do
add i to the label of t (at the end)

In the order σ obtained by the algorithm on an arbitrary graph G, we will always denote by w the vertex firstly chosen (i.e. $\sigma(n)$) and by z the vertex chosen in the end (i.e. $\sigma(1)$).

In [12] and [10] we identified certain properties of the order obtained by applying LexBFS on an arbitrary graph. For the sake of completeness, we give them again here below, without proof. To this end, suppose our graph has at least two vertices and define a sequence of sets H_i, H_i' $(i\geqslant 0)$ of V as follows (see Fig. 1). The notation $N_X(q)$, for $X\subseteq V$ and $q\in V$, designates the set of neighbours of q situated in X. When $X=V$ we simply write $N(q)$.

Step 1. Let $H_0=\{w\}$ and $i=0$.
Step 2. Define $H_i'=\{h'\in V \mid \text{for all}\, h\in H_i, \sigma^{-1}(h')<\sigma^{-1}(h)\}$.
Step 3. Partition H_i into
$T_i=H_i\cap N(z)$;
$U_i=\{h\in H_i-T_i \mid N_{H_i'}(h)\neq\emptyset\}$
$V_i=H_i-(T_i\cup U_i)$
Step 4. If $U_i=\emptyset$ then STOP.
Step 5. Define $H_{i+1}=\bigcup_{u\in U_i} N_{H_i'}(u)$; $i:=i+1$; goto Step 2.

We easily have that $H_0\subseteq V-\{z\}$, and then we can obtain by induction that $H_i\subseteq V-\{z\}$ $(i>0)$. Calling an *interval* any set $R\subseteq V$ such that every $u\in V-R$ either satisfies $\sigma^{-1}(r)<\sigma^{-1}(u)$ for every $r\in R$, or satisfies $\sigma^{-1}(u)<\sigma^{-1}(r)$ for every $r\in R$, the following properties hold (see [12]):

Claim 1. *For every graph* $G=(V,E)$ *we have* (*assuming that* $H_{i+1}\neq\emptyset$):

(P1) *for all* i ($i\geqslant 0$), *the set* H_{i+1} *is an interval and* $H_{i+1}\subseteq H_i'$ (*so that* $H_{i+1}'\subset H_i'$);
(P2) *for all* i ($i\geqslant 0$), *the set* $H_i\cup H_{i+1}$ *is an interval*;
(P3) *for all* i ($i\geqslant 0$), *every* $h'\in H_{i+1}'$ *has* $N(h')\subseteq H_{i+1}'\cup H_{i+1}\cup T_i\cup T_{i-1}\cup\cdots\cup T_0$.

Notice that (P1), (P2) imply that for an arbitrary index i considered in Step 5, $H_0\cup H_1\cup\cdots\cup H_i\cup H_i'$ is a partition of V. Then the sequence of sets must be finite, so there exists an index p such that $U_p=\emptyset$ (and $H_p'\neq\emptyset$ since $z\in H_p'$). This is precisely the index p which appears in the claim below (proved in [10]) and in the rest of the paper.

As usual, the parity of a path is *odd* if it has odd number of edges, and *even* if it has even number of edges. If x,y are (not necessarily distinct) vertices in G, the notation $G-xy$ designates the graph obtained by removing, if it exists, the edge xy. A *module* of a graph $G=(V,E)$ is an induced subgraph $H=(V(H),E(H))$ of G such that every vertex in $V-V(H)$ is either adjacent to every vertex in H or to no vertex in H.

In order to make our presentation homogeneous, in the claim below and in the rest of the paper we make the following convention: if x,y are two letters which denote the *same* vertex, then a *chordless path joining* x *to* y is simply a chordless cycle containing the vertex $x=y$.

Claim 2. *For every graph* $G=(V,E)$ *we have*:

(P4) *for all* i ($0\leqslant i\leqslant p-1$), *every vertex in* T_i *is adjacent to every vertex of* H_{i+1}';
(P5) *every vertex in* T_p *is adjacent to every vertex of* H_p';
(P6) H_p' *is a module of* G, *and its neighbourhood in* $V-H_p'$ *is* $T_0\cup T_1\cup\cdots\cup T_p$.
(P7) *for all* i ($1\leqslant i\leqslant p$) *if* x *and* y *are distinct vertices in* H_i, *there exists a chordless path joining* x,y *whose internal vertices are in* $H_0\cup H_1\cup\cdots\cup H_{i-1}-N(H_i')$.
(P8) *for all* i ($1\leqslant i\leqslant p$), *if* G *is a Berge graph and* x,y *are vertices in* H_i, *then all the chordless paths with the properties below have the same parity*:
  (a) *they join* x *to* y *in* $G-xy$
  (b) *their internal vertices are in* $H_i'=H_{i+1}\cup\cdots\cup H_p\cup H_p'$
  (c) *their length is at least* 2 (*if* x,y *are distinct and non-adjacent*), *respectively at least* 3 (*if* x,y *are distinct and adjacent*), *respectively*, *at least* 4 (*if* x,y *are not distinct*)

Remark 1. The different minimum lengths in condition (c) insure that, if two chordless paths with properties (a)–(c) and different parities existed, then two chordless cycles of length at least four and of different parities could be found (a contradiction would then be obtained, since at least one of the cycles would be a C_{2k+1} for a suitable k). The chordless cycles would be obtained as follows: if x,y are distinct and non-adjacent, it is sufficient to put together each of the two paths in (P8) and the path in (P7); if x,y are distinct and adjacent, it is sufficient to add the edge xy to each of the paths in (P8); finally, if x,y are not distinct, each of the two paths in (P8) is in fact a chordless cycle containing $x=y$.

3. Main result

Let G be a connected C_4-free Berge graph. Assume we have applied to G the algorithm LexBFS and we have obtained the decomposition above. The entire notation (for w and z included) is preserved. We can assume that $T_0=\emptyset$, for otherwise w is a dominating vertex and the main result presented in this section easily follows by induction. The reasoning below uses a number of claims whose (sometimes long) proofs are given in the next section.

We will prove the following result:

Theorem 1. *Let $G=(V,E)$ be a C_4-free Berge graph, whose last vertex numbered by the LexBFS algorithm is z. Then the neighbourhood $N(z)$ of z induces a P_4-free graph, or else G is breakable.*

Proof of Theorem 1. The proof will be done by induction on the number of vertices in G. When this number is small, Theorem 1 obviously holds. Moreover, we can assume that $H'_p=\{z\}$; otherwise, by (P6), H'_p is a module of cardinality 2 or more, and $\{z\}\cup N_{V-H'_p}(z)$ is a star-cutset of G (we have assumed that $p>1$, so that w is non-adjacent to z).

All along the following proofs we will use the observation, deduced from property (P4), that for indices i,j such that $|i-j|\geqslant 2$ we have all possible edges between T_i and T_j. $\square$

Lemma 1. *If T_i,T_j $(i<j)$ are non-empty, then $j<i+4$ or G is breakable.*

Proof of Lemma 1. In fact, we will show a little stronger property: either $j<i+4$, or there exists a vertex $u\in N(z)$ which dominates z (i.e. $\{u\}\cup N(u)\supseteq N(z)$). By contradiction, assume none of the two statements holds. Then no vertex u as described exists, but we can find two non-empty sets T_i,T_j such that $i+4\leqslant j$ (without loss of generality, take the minimum index i and the maximum index j with this property).

By Property (P4) in Claim 2, every vertex in T_i is adjacent to every vertex of $T_{i+2}\cup T_{i+3}\cup\cdots\cup T_j$ and by symmetry, every vertex in T_j is adjacent to every vertex of $T_i\cup T_{i+1}\cup\cdots\cup T_{j-2}$.

As no vertex in $N(z)$ dominates z, none of the sets $T_i\cup T_{i+1}$, $T_j\cup T_{j-1}$ induces a clique. Otherwise, either a vertex in T_i or a vertex in T_j dominates z (by (P6) and (P4)). Thus, there exist two non-adjacent vertices v_1, v_2 belonging to $T_i\cup T_{i+1}$ and two other non-adjacent vertices v_3, v_4 belonging to $T_j\cup T_{j-1}$. But then v_1, v_2, v_3 and v_4 induce a C_4, a contradiction. $\square$

If G is breakable, then Theorem 1 is proved. Otherwise, $j<i+4$ and there exists a unique i_0 $(1\leqslant i_0\leqslant p)$ such that $T_{i_0}\neq\emptyset$ and $T_j=\emptyset$ for every $j\in\{1,2,\ldots,p\}-\{i_0,i_0+1,i_0+2,i_0+3\}$. We can strengthen this result as follows:

Lemma 2. *We have $i_0\geqslant p-3$, or G is breakable.*

Proof of Lemma 2. Assume that $i_0 < p-3$, and let us show that G is breakable. We deduce that $T_p = \emptyset$ (since $p \notin \{i_0, i_0+1, i_0+2, i_0+3\}$) and, because of $U_p = \emptyset$ and $H_p \neq \emptyset$, we deduce that $V_p \neq \emptyset$. Let $x \in V_p$ and notice that, by the definition of V_p, $xz \notin E$. Now, property (P4) for T_{i_0}, T_{i_0+1} and V_p implies that every vertex in $T_{i_0} \cup T_{i_0+1}$ is adjacent to both x and z, so $T_{i_0} \cup T_{i_0+1}$ induces a clique (otherwise two non-adjacent vertices of this set and x, z form a C_4).

On the other hand, again by (P4) for i_0 we have that every $t \in T_{i_0}$ is adjacent to every $t' \in T_s$, $s \geqslant i_0 + 2$. Then every vertex $u \in T_{i_0}$ dominates z (and such an u exists since $T_{i_0} \neq \emptyset$). Thus, G is breakable. □

We can even go further in counting the non-empty sets T_i:

Lemma 3. *We have $i_0 \geqslant p-2$, or G is breakable.*

Proof of Lemma 3. Assume that $T_{p-3} \neq \emptyset$, and let us show that G is breakable.

Let us first notice that $T_p \neq \emptyset$. Indeed, in the contrary case, as $H_p \neq \emptyset$ and $U_p = \emptyset$, we have that $V_p \neq \emptyset$. Let $u \in V_p$. By (P4), every vertex in $T_{p-3} \cup T_{p-2}$ is adjacent to u, z, so $T_{p-3} \cup T_{p-2}$ must induce a clique (otherwise a C_4 may be built with u, z and two non-adjacent vertices in $T_{p-3} \cup T_{p-2}$). But then T_{p-3} is completely adjacent to T_{p-2}, T_{p-1} and every vertex of T_{p-3} dominates z. Consequently, G is breakable.

Now, $T_{p-3} \neq \emptyset$ and $T_p \neq \emptyset$. Since T_{p-3} cannot be completely adjacent to $T_{p-2} \cup T_{p-1} \cup T_p$ (in this case, as before, G is breakable and the lemma is proved) and by (P4) we already have that T_{p-3} is adjacent to $T_{p-1} \cup T_p$, there must exist non-adjacent vertices $a \in T_{p-2}$, $c \in T_{p-3}$. For the same reasons, there must exist non-adjacent vertices $d \in T_{p-1}$, $b \in T_p$. Then $[a, b, c, d]$ is a P_4 ($ad \notin E$, else $[a, b, c, d]$ is a C_4, a contradiction).

Since $H_{p-1} = N_{H'_{p-2}}(U_{p-2})$, there exists $d'' \in U_{p-2}$ such that $d''d \in E$. Similarly, there exist $a', d' \in U_{p-3}$ such that $aa' \in E$, $d''d' \in E$. By the definition of U_i ($i \geqslant p-3$), $a'z, d'z, d''z \notin E$. Moreover, $a'd, a'b, d'b, d''b, dd' \notin E$ since the vertices in U_i are non-adjacent to the vertices in H'_{i+1}.

Denote
$A_1 = N_{U_{p-4}}(a')$ and
$D_1 = N_{U_{p-4}}(d')$.
Because of $H_{p-3} = N_{H'_{p-4}}(U_{p-4})$, these sets are non-empty (notice that $H_{p-3} \neq H_0$ since $T_{p-3} \neq \emptyset$, while $T_0 = \emptyset$). An arbitrary vertex of each set A_1, D_1 is denoted a_1, d_1 respectively. Note that, by the definition of U_i, a_1 and d_1 are non-adjacent to the vertices z, b, d, a, d''.

We will prove that d is dominated by c. We successively have:

- $a'c \notin E$; otherwise $[a, a', c, z, a]$ is a C_4.
- $d''a \notin E$; otherwise $[d'', a, z, d, d'']$ is a C_4.
- $a'd'' \notin E$; otherwise $[a, a', d'', d, z, a]$ is a C_5.
- $d'a \notin E$; otherwise $[d', a, z, d, d'', d']$ is a C_5.
- $a_1c \notin E$; otherwise $[a_1, c, b, a, a', a_1]$ is a C_5.

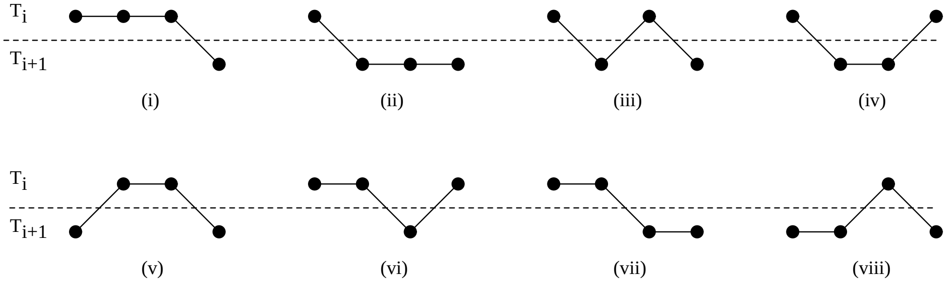

Fig. 2. The eight possibilities for a P_4 with vertices in both T_i and T_{i+1}.

- $d''c \in E$; otherwise $cd' \notin E$ (else $[c, d', d'', d, c]$ is a C_4), $d'a' \notin E$ (else $[d', a', a, b, c, d, d'', d']$ is a C_7), $cd_1 \notin E$ (else $[c, d_1, d', d'', d, c]$ is a C_5) and (P8) is contradicted by the pair of paths $[a_1, a', a, b, c, d, d'', d', d_1]$ and $[a_1, a', a, z, d, d'', d', d_1]$.
- $N_{H_{p-2}}(d) \subseteq N(c)$; indeed, $N_{H_{p-2}}(d) = N_{U_{p-2}}(d) \cup N_{T_{p-2}}(d)$, and we both have $N_{U_{p-2}}(d) \subseteq N(c)$ (since d'' was arbitrarily chosen in $N_{U_{p-2}}(d)$ and $d''c \in E$), $N_{T_{p-2}}(d) \subseteq N(c)$ (since otherwise a counterexample t satisfies $tb \in E$, by (P4), so $[t, d, c, b, t]$ is a C_4).
- $N_{H_{p-3}}(d) \subseteq N(c)$; indeed, $N_{H_{p-3}}(d) = N_{T_{p-3}}(d)$ and every $t \in N_{H_{p-3}}(d)$ satisfies $tb \in E$ (by (P4)), and $tc \in E$ (otherwise $[b, c, d, t, b]$ is a C_4).
- $N(d) \subseteq N(c)$; indeed, $N(d) \subseteq T_1 \cup T_2 \cup \cdots \cup T_{p-4} \cup T_{p-3} \cup H_{p-2} \cup H_{p-1} \cup H_p \cup \{z\}$. Now, $T_1 \cup \cdots \cup T_{p-4} = \emptyset$ (by Lemma 2), $N_{H_{p-2}}(d) \cup N_{H_{p-3}}(d) \subseteq N(c)$ (as proved above), and $H_{p-1} \cup H_p \cup \{z\} \subseteq N(c)$, by (P4). $\square$

By this lemma, either $T_{p-3} = \emptyset$ or G is breakable. In the rest of the paper we will assume that $T_{p-3} = \emptyset$ (otherwise Theorem 1 is proved).

Now, we continue the proof of Theorem 1 by showing that a P_4 induced in $N(z)$ cannot be contained in one set T_i ($i \geqslant p-2$), then that it cannot be contained in two such sets, and finally that it cannot be contained in three such sets. In each step, we consider all the possible configurations and show that if one of them appears in the graph, a contradiction can be found (or the graph is breakable). This yields a long (although not very complicated) proof. For some of the intermediate results, the proofs may be found in [10], the others are in Section 4.

Lemma 4. *For every i ($p-2 \leqslant i \leqslant p$), T_i is P_4-free.*

Proof of Lemma 4. Can be found in [10]. $\square$

Lemma 5. *For every i ($p-2 \leqslant i \leqslant p-1$), $T_i \cup T_{i+1}$ is P_4-free, or G is breakable.*

Proof of Lemma 5. By an exhaustive search, we can find that (up to a symmetry), the only possible configurations for a P_4 denoted $[a, b, c, d]$ which would be included in $T_i \cup T_{i+1}$ are (see Fig. 2):

(i) $a, b, c \in T_i$ and $d \in T_{i+1}$;
(ii) $a \in T_i$ and $b, c, d \in T_{i+1}$;

(iii) $a, c \in T_i$ and $b, d \in T_{i+1}$;
(iv) $a, d \in T_i$ and $b, c \in T_{i+1}$;
(v) $b, c \in T_i$ and $a, d \in T_{i+1}$;
(vi) $a, b, d \in T_i$ and $c \in T_{i+1}$;
(vii) $a, b \in T_i$ and $c, d \in T_{i+1}$;
(viii) $c \in T_i$ and $a, b, d \in T_{i+1}$.

In [10] it is shown that the configurations (i)–(v) cannot appear. To prove that (vi) and (vii) imply that G is breakable, we use the following claim, in which we also include a configuration that we will need later.

Claim 3. *Let $[a, b, c, d]$ be a P_4 in G such that $a, b \in T_i$, $c \in T_{i+1}$ and $d \in T_{i-1} \cup T_i \cup T_{i+1}$. Then $S = \{b\} \cup N(b) - \{a\}$ is a star-cutset of G.*

To finish the proof of Lemma 5, it remains to show that a P_4 of type (viii) cannot appear. We use the following claim in which, once again, we treat together the case in which we are immediately interested, and a case that will be needed later.

Claim 4. *No P_4 $[a, b, c, d]$ in G has $c \in T_i$, $a, b \in T_{i+1}$ and $d \in T_{i+1} \cup T_{i+2}$, or G is breakable.*

Now, Lemma 5 is proved. □

To finish the proof of Theorem 1, we show that if the graph induced by $N(z) = T_{p-2} \cup T_{p-1} \cup T_p$ contains a P_4, then G is breakable.

By performing an exhaustive search (taking into account that every vertex in T_{p-2} is adjacent to every vertex in T_p), we find that (up to symmetries) the only possibilities for a P_4 $[a, b, c, d]$ to have vertices in T_{p-2}, in T_{p-1} and in T_p are the following (see Fig. 3):

(ix) $d \in T_{p-2}$, $a, b \in T_{p-1}$ and $c \in T_p$;
(x) $c \in T_{p-2}$, $a, b \in T_{p-1}$ and $d \in T_p$;
(xi) $c \in T_{p-2}$, $a \in T_{p-1}$ and $b, d \in T_p$;
(xii) $b \in T_{p-2}$, $a, d \in T_{p-1}$ and $c \in T_p$;
(xiii) $b, d \in T_{p-2}$, $a \in T_{p-1}$ and $c \in T_p$.

By Claim 3 (case $d \in T_{i-1}$) for $i = p - 1$, (ix) cannot appear. By Claim 4 (case $d \in T_{i+2}$) for $i = p - 2$, we have that (x) cannot appear.

To show that (xi) cannot appear we use the following claim:

Claim 5. *No P_4 $[a, b, c, d]$ in G is of type* (xi).

Then, we solve cases (xii) and (xiii):

Claim 6. *No P_4 $[a, b, c, d]$ has $b \in T_{p-2}$, $a \in T_{p-1}$, $c \in T_p$ and $d \in T_{p-2} \cup T_{p-1}$, or G has star-cutset $S = \{b\} \cup N(b) - \{a\}$.*

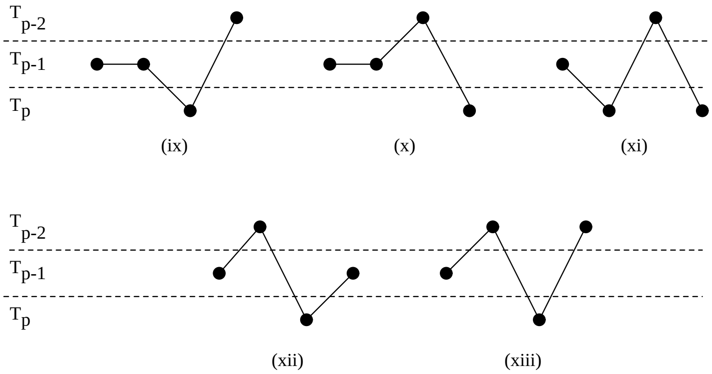

Fig. 3. The five possibilities for a P_4 with vertices in T_{p-2}, in T_{p-1} and in T_p.

The sketch of the proof is now finished. In the next section we give the proofs of the claims. □

4. Proofs

In any of the claims we have to prove here, the proofs use simple arguments and a large number of small results. We precede each such small result by a •, and we give the proof in the same paragraph. In this way, the reader will be able to quickly identify each small result.

The proofs are done by contradiction, and the contradiction is obtained either by indicating two paths that contradict (P8), or by indicating two chordless cycles of different parities (and length at least 4), one of which must be odd. We recall that, with the convention we made for (P8), cycles will sometimes be found in (P8) instead of paths.

In the remaining of this section, we will define sets A_1, B_1, C_1, D_1 which are always non-empty. The notation a_1 (respectively b_1, c_1, d_1) will then concern an arbitrary (but fixed) vertex in A_1 (respectively B_1, C_1, D_1). The notation $(ab)_1$ will concern an arbitrary (but fixed) vertex in $A_1 \cap B_1$, whenever this set is (or is supposed) non-empty (and similarly for the other intersections). The notation $(a.b)_1$ will concern an arbitrary (but fixed) vertex in $A_1 - B_1$, whenever this set is (or is supposed) non-empty (and similarly for the other set differences).

A path will be denoted by $[v_1, v_2, \dots, v_k]$ and a cycle by $[v_1, v_2, \dots, v_k, v_1]$. The sub-path $[v_i, v_{i+1}, \dots, v_{k-1}, v_k]$ of a path or cycle will be denoted $P_{v_i v_k}$. For two sets of vertices R, Q, we denote $N_R(Q) = \bigcup_{q \in Q} N_R(q)$; when $R = V - Q$, we simply write $N(Q)$.

Proof of Claim 3. We will deal with the three cases $d \in T_{i-1}$, $d \in T_i$, $d \in T_{i+1}$ in parallel, and we will call them case 1, 2, 3 for short. The idea of the proof is to define a set X, to assume by contradiction that X is still connected to the rest of the graph when

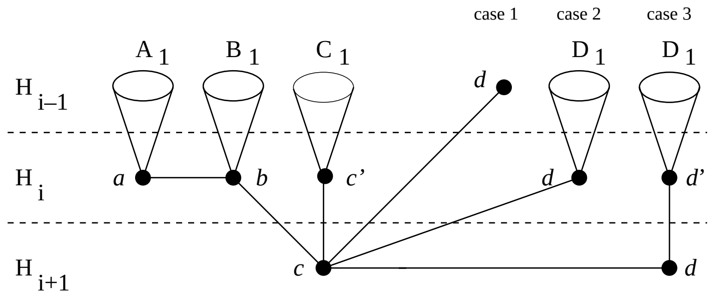

Fig. 4. The sets A_1, B_1, C_1, D_1 in the three cases with respect to d.

$\{b\} \cup N(b) - \{a\}$ is removed, and then to try to contradict property (P8) by building chordless paths of different parity between two vertices situated in H_{i-1}.

Because of $H_{i+1} = N_{H_i'}(U_i)$, there exists $c' \in U_i$ such that $c'c \in E$. In case 3, for the same reason there exists $d' \in U_i$ such that $d'd \in E$. Denote (see Fig. 4)

$A_1 = N_{U_{i-1}}(a)$,
$B_1 = N_{U_{i-1}}(b)$,
$C_1 = N_{U_{i-1}}(c')$ and
$D_1 = N_{U_{i-1}}(d)$ (case 2), respectively, $D_1 = N_{U_{i-1}}(d')$ (case 3).

Because of $H_i = N_{H_{i-1}'}(U_{i-1})$, these sets are non-empty and every vertex of $A_1 \cup B_1 \cup C_1 \cup D_1$ is non-adjacent to all the vertices in H_i'. We have:

- $c'a \notin E$; otherwise $[a, c', c, z, a]$ is a C_4.
- $d'a \notin E$ (case 3); otherwise $[a, d', d, z, a]$ is a C_4.
- $d'b \notin E$ (case 3); otherwise $[b, d', d, z, b]$ is a C_4.
- $A_1 \cap D_1 = \emptyset$ (cases 2, 3); otherwise $[a, (ad)_1, d, z, a]$ is a C_4 (case 2), respectively $[a, (ad)_1, d', d, z, a]$ is a C_5 (case 3).
- $B_1 \cap D_1 = \emptyset$ (cases 2, 3); otherwise $[b, (bd)_1, d, z, b]$ is a C_4 (case 2), respectively $[b, (bd)_1, d', d, z, b]$ is a C_5 (case 3).
- $A_1 \cap C_1 = \emptyset$; otherwise $[a, (ac)_1, c', c, z, a]$ is a C_5.
- $A_1 \subseteq B_1$; otherwise (P8) is contradicted by the pair of paths $[(a.b)_1, a, b, c, d]$ and $[(a.b)_1, a, z, d]$ (case 1), respectively $[(a.b)_1, a, b, c, d, d_1]$ and $[(a.b)_1, a, z, d, d_1]$ (case 2), respectively $[(a.b)_1, a, b, c, d, d', d_1]$ and $[(a.b)_1, a, z, d, d', d_1]$ (case 3).
- $c'b \in E$, since in the contrary case we have $C_1 \cap B_1 = \emptyset$ (otherwise $[b, (cb)_1, c', c, b]$ is a C_4) and then (P8) is contradicted by the pair of paths $[a_1, a, z, c, c', c_1]$ and $[a_1, b, c, c', c_1]$.
- $c'd \notin E$; otherwise $[b, c', d, z, b]$ is a C_4.

Remark 2. Since c' is arbitrarily chosen in U_i such that $c'c \in E$, the reasoning we made for c' is valid for every $y \in U_i$ such that $yc \in E$; we can deduce then that $yd \notin E$.

- $cd' \notin E$ (case 3); follows from Remark 2 with $y = d'$.
- $c'd' \notin E$ (case 3); otherwise $[c', d', d, c, c']$ is a C_4.

- $C_1 \cap D_1 = \emptyset$ (cases 2, 3); otherwise $[c', (cd_1), d, c, c']$ is a C_4 (case 2), respectively $[c', (cd)_1, d', d, c, c']$ is a C_5 (case 3).
- $C_1 \subseteq B_1$; otherwise (P8) is contradicted by the pair of paths $[d, z, b, c', (c.b)_1]$ and $[d, c, c', (c.b)_1]$ (case 1), respectively $[d_1, d, z, b, c', (c.b)_1]$ and $[d_1, d, c, c', (c.b)_1]$ (case 2), respectively $[d_1, d', d, z, b, c', (c.b)_1]$ and $[d_1, d', d, c, c', (c.b)_1]$ (case 3).
- A_1, C_1 are joined by all possible edges; otherwise, if $a_1c_1 \notin E$, the paths $[a_1, b, c_1]$ and $[a_1, a, z, c, c', c_1]$ contradict (P8).

Let X be the connected component containing a in the subgraph induced by $U_i \cup V_i \cup U_{i+1} \cup V_{i+1} - S$ (recall that $S = \{b\} \cup N(b) - \{a\}$). We will show that, in $G - S$, the set X is disconnected from the rest of the graph, so that S is a star-cutset of G.

By the definition of X, for every $x \in X - \{a\}$ we have $xb \notin E$ and $xz \notin E$.

Remark 3. If we denote $X_i = X \cap H_i$, $X_{i+1} = X \cap H_{i+1}$, then for every $x \in X_i$ we have $xc \notin E$ (since then $x \in U_i$ and we conclude by Remark 2). So $xc \in E$ implies $x \in X_{i+1}$.

Remark 4. If $y \in X$ such that $yc \in E$ or $yc' \in E$, then $ya \notin E$ (otherwise $[y, a, b, c, y]$, respectively $[y, a, b, c', y]$, is a C_4).

Now, consider some $x \in X - \{a\}$ and observe that we have at least one of the following situations:

I. there exists a chordless path P_{xa} joining x to a in X, such that every vertex $y \in V(P_{xa})$ satisfies $yc \notin E$ or $yc' \notin E$;

II. there exists a chordless path $P_{xx'}$ (contained in X) joining x to some $x' \in X - \{a\}$, such that $x'c, x'c' \in E$, $x'a \notin E$ and every vertex $y \in V(P_{xx'}) - \{x'\}$ satisfies $ya \notin E$, and $yc \notin E$ or $yc' \notin E$ (the particular case $x = x'$ is accepted).

Indeed, if I is not true, then on every chordless path P_{xa} joining x to a in X, we can find some vertex p for which $pc \in E$ and $pc' \in E$. Thus if we take the vertex p with these properties which is closest to x, we have II (by Remark 4, x' is not adjacent to a). Before showing that $N(X) \subseteq S$, we establish two properties of the paths in I, respectively, in II.

Property I. For every $y \in V(P_{xa})$ we have $yc \notin E, yd \notin E$, $yd' \notin E$ (case 3), $yd_1 \notin E$ (cases 2, 3; as usual, $d_1 \in D_1$).

It is sufficient to show that for every $y \in V(P_{xa})$ we have $yc \notin E, yd \notin E$, since then

- $yd' \notin E$ (case 3); otherwise, consider $y \in V(P_{xa})$ (with $yd' \in E$) as close as possible to a and notice that the cycles $[P_{ya}, b, c, d, d', y]$ and $[P_{ya}, z, d, d', y]$ are chordless and of different parities (thus one of them is odd, a contradiction).
- $yd_1 \notin E$ (cases 2, 3); otherwise, consider $y \in V(P_{xa})$ (with $yd_1 \in E$) as close as possible to a and notice that the chordless cycles below have different parities: $[P_{ya}, b, c, d, d_1, y]$ and $[P_{ya}, z, d, d_1, y]$ (case 2), respectively $[P_{ya}, b, c, d, d', d_1, y]$ and $[P_{ya}, z, d, d', d_1, y]$ (case 3).

Thus we only have to show that for every $y \in V(P_{xa})$ we have $yc \notin E, yd \notin E$. If this is not the case, let $y \in V(P_{xa})$ such that $yc \in E$ or $yd \in E$, and such that y is as close as possible to a along P. Then we have:

- $yc \in E$; otherwise $yd \in E$ and the cycles $[P_{ya}, b, c, d, y]$, $[P_{ya}, z, d, y]$ are chordless and of different parities, a contradiction.
- $yc' \notin E$ since $yc \in E$ and since we are in case I.
- $y \in X_{i+1}$; by Remark 3, since $yc \in E$.
- $yd_1 \notin E$, since $y \in X_{i+1}$ and $d_1 \in U_{i-1}$.
- $pd' \notin E$, for every $p \in V(P_{ya}) - \{y\}$ (case 3); the proof is identical to the one for $yd' \notin E$ (see above).
- $pd_1 \notin E$, for every $p \in V(P_{ya}) - \{y\}$ (cases 2, 3); the proof is identical to the one for $yd_1 \notin E$ (see above).
- $pc' \notin E$, for every $p \in V(P_{ya}) - \{y\}$; otherwise, if we consider p as close as possible to a such that $pc' \in E$, the chordless cycles $[P_{pa}, b, c', p]$ and $[P_{pa}, z, c, c', p]$ are of different parities, a contradiction.
- P_{ya} is odd; otherwise the chordless cycle $[P_{ya}, b, c, y]$ is odd.
- $pc_1 \notin E$, for every $p \in V(P_{ya})$; in the contrary case, let p be as close as possible to a which contradicts the affirmation and notice that we successively have: $p \neq y$ (since $y \in X_{i+1} \subseteq H_{i+1}$, so y has no neighbour in U_{i-1}), P_{pa} is odd (since the chordless cycle $[P_{pa}, b, c_1, p]$ has to be even), so (P8) is contradicted by the pair of paths $[c_1, b, z, d]$ and $[c_1, P_{pa}, z, d]$ (case 1), respectively $[c_1, b, z, d, d_1]$ and $[c_1, P_{pa}, z, d, d_1]$ (case 2), respectively $[c_1, b, z, d, d', d_1]$ and $[c_1, P_{pa}, z, d, d', d_1]$ (case 3).
- $pa_1 \notin E$, for every $p \in V(P_{ya})$; in the contrary case, let p be as close as possible to y which contradicts the affirmation and notice that we successively have: $p \neq y$ (since $y \in X_{i+1} \subseteq H_{i+1}$, so y has no neighbour in U_{i-1}), P_{yp} is even (the chordless cycle $[P_{yp}, a_1, b, c, y]$ has to be even), and then the chordless cycle $[P_{yp}, a_1, c_1, c', c, y]$ is odd.
- case 1 is finished, since (P8) is contradicted by the pair of paths $[a_1, P_{ay}, c, d]$ and $[a_1, a, z, d]$.
- $yd \in E$ (cases 2, 3); otherwise $yd' \notin E$ (case 3; else $[y, d', d, c, y]$ is a C_4), $yd_1 \notin E$ (cases 2, 3; otherwise $[y, d_1, d, c, y]$ is a C_4 in case 2 and $[y, d_1, d', d, c, y]$ is a C_5 in case 3) and (P8) is contradicted by the pair of paths $[a_1, P_{ay}, c, d, d_1]$ and $[a_1, a, z, d, d_1]$ (case 2), respectively $[a_1, P_{ay}, c, d, d', d_1]$ and $[a_1, a, z, d, d', d_1]$ (case 3).
- $yd' \notin E$ (case 3); otherwise (P8) is contradicted by the pair of paths $[a_1, b, c, d, d', d_1]$ and $[a_1, P_{ay}, d', d_1]$.

But now (P8) is contradicted by the pair of paths $[c_1, b, P_{ay}, d, d_1]$ and $[c_1, b, z, d, d_1]$ (case 2), respectively $[c_1, b, P_{ay}, d, d', d_1]$ and $[c_1, b, z, d, d', d_1]$ (case 3).

Property II. For every $y \in V(P_{xx'})$ we have $yd \notin E$.

If the property above does not hold, there exists some $y \in V(P_{xx'})$ such that $yd \in E$. We have:

- $x'd \notin E$; otherwise $[x', c', b, z, d, x']$ is a C_5.
- $x' \in X_{i+1}$ because of Remark 3.

- $pa_1 \notin E$, for every $p \in V(P_{yx'})$; otherwise let p be the counterexample closest to x' and let x'' be the vertex of $P_{px'}$ closest to p such that $x''c \in E$ (such a vertex exists since $x'c \in E$). Then the chordless cycles $[a_1, a, z, c, P_{x''p}, a_1]$ and $[a_1, b, c, P_{x''p}, a_1]$ have different parities.
- $pc_1 \notin E$, for every $p \in V(P_{yx'})$; otherwise, consider the vertex p with $pc_1 \in E$ which is closest to y and notice that the chordless cycles $[p, c_1, a_1, a, z, d, P_{yp}]$ and $[p, c_1, b, z, d, P_{yp}]$ have different parities.

Now, denote by x'' the vertex of $P_{x'y}$ which is closest to y such that $x''c' \in E$. We have:

- $P_{x''y}$ is odd, since the cycle $[c', b, z, d, P_{yx''}, c']$ has to be even.
- case 1 is finished, since (P8) is contradicted by the chordless paths $[d, z, b, c_1]$ and $[d, P_{yx''}, c', c_1]$.

  Define the non-empty (because of $H_{i-1} = N_{H'_{i-2}}(U_{i-2})$) sets:

  $A_2 = N_{U_{i-2}}(A_1)$,
  $C_2 = N_{U_{i-2}}(C_1)$ and
  $D_2 = N_{U_{i-2}}(D_1)$.

Notice that $H_{i-1} \neq H_0$ since H_{i-1} has at least two distinct vertices, a_1 and c_1. The convention of notation we established for A_1, B_1, C_1, D_1 is extended to A_2, C_2, D_2. Moreover, if $a_2 \in A_2$, then $a_1 \in A_1$ denotes a neighbour of a_2 (and similarly for C_2, D_2).

- $A_2 \subseteq C_2$; otherwise, (P8) is contradicted by the pair of paths $[(a.c)_2, a_1, b, z, d, d_1, d_2]$ and $[(a.c)_2, a_1, c_1, c', c, d, d_1, d_2]$ (case 2), respectively $[(a.c)_2, a_1, b, z, d, d', d_1, d_2]$ and $[(a.c)_2, a_1, c_1, c', c, d, d', d_1, d_2]$ (case 3).

But now, if we take a chordless path $P'_{x''d_2}$ joining x'' and d_2 in the graph induced by the vertices in $V(P_{x''y}) \cup \{d, d', d_1, d_2\}$) ($d'$ must be forgotten if we are in case 2), then (P8) is contradicted by the pair of paths $[a_2, a_1, b, c', P'_{x''d_2}]$ and $[a_2, c_1, c', P'_{x''d_2}]$.

Remark 5. The reasoning we made to prove Property II is valid for every vertex $u \in T_{i-1} \cup T_i \cup T_{i+1}$ such that $uc \in E$, $ub \notin E$, since d has all these properties and was arbitrarily chosen. So Property II is true not only for d, but for any u as described before.

Now, Properties I and II are proved and we can show that $N(X) \subseteq N(b)$ (and this easily implies that X is disconnected in $G - S$ from the rest of the graph, which is non-empty since it contains d). Notice that:

- $N(X) - N(b) \subseteq U_{i-1} \cup T$; by the definition of X and the properties (P3), (P4) we have that $N(X) \subseteq U_{i-1} \cup U_i \cup V_i \cup U_{i+1} \cup V_{i+1} \cup T \cup \{z\}$. But $z \in N(b)$ and every neighbour of X in $U_i \cup V_i \cup U_{i+1} \cup V_{i+1}$ must be in $N(b)$ too (otherwise it should have been put in X, a contradiction).

We show that $N_{U_{i-1}}(X) \subseteq N(b)$.

Suppose this is not the case and let $u \in N_{U_{i-1}}(X) - N(b)$. Then $ub \notin E$ and u has at least one neighbour $x \in X$. We have:

- $ua \notin E$; otherwise $u \in A_1 - B_1$, a contradiction to $A_1 \subseteq B_1$.
- $uc' \notin E$; otherwise $u \in C_1 - B_1$, a contradiction to $C_1 \subseteq B_1$.
- x is in situation II; otherwise, let P_{xa} be the path indicated in I and assume that u has no neighbour but x on this path (else, we take the neighbour of u which is closest to a on this path, and we rename it x). Then we successively have $uc \notin E$ (since $u \in U_{i-1}$ and $c \in T_{i+1}$), $ud \notin E$ (else the chordless cycles $[x,u,d,c,b,P_{ax}]$ and $[x,u,d,z,P_{ax}]$ have different parities), $ud' \notin E$ (case 3; otherwise the cycles $[x,u,d',d,c,b,P_{ax}]$ and $[x,u,d',d,z,P_{ax}]$ are of different parities), so (P8) is contradicted by the paths $[u,P_{xa},b,c,d]$ and $[u,P_{xa},z,d]$ (case 1), respectively $[u,P_{xa},b,c,d,d_1]$ and $[u,P_{xa},z,d,d_1]$ (case 2), respectively $[u,P_{xa},b,c,d,d',d_1]$ and $[u,P_{xa},z,d,d',d_1]$ (case 3).

Then let $P_{xx'}$ be the path in II and assume that x is the neighbour of u which is closest to x' on the path (otherwise we can perform, as before, a change of notation). Furthermore, let y be the vertex of $P_{xx'}$ which is adjacent to c or c' and is as close as possible to x (such a vertex exists since x' is one of the candidates; possibly, x' is not the closest one).

- $yc \notin E$, since in the contrary case we have $pa_1 \notin E$ for every $p \in V(P_{yx})$ (otherwise the counterexample p which is closest to y insures that the chordless cycles $[a_1,a,z,c,P_{yp},a_1]$ and $[a_1,b,c,P_{yp},a_1]$ are of different parity), and then the paths $[u,P_{xy},c,b,a_1]$, $[u,P_{xy},c,z,a,a_1]$ are chordless and of different parities.
- $yc' \in E$ (and $yc \notin E$); follows immediately from the choice of y.
- $pd \notin E$, for every $p \in V(P_{xy})$; by Property II.
- $pd' \notin E$, for every $p \in V(P_{xy})$ (case 3); otherwise the counterexample p which is closest to y implies that the chordless cycles $[d',d,c,c',P_{yp},d']$ and $[d',d,z,b,c',P_{yp},d']$ are of different parities.
- $pd_1 \notin E$, for every $p \in V(P_{xy})$ (cases 2,3); otherwise the counterexample p which is closest to y implies that the following chordless cycles are of different parities: $[d_1,d,c,c',P_{yp},d_1]$ and $[d_1,d,z,b,c',P_{yp},d_1]$ (case 2), respectively $[d_1,d',d,c,c',P_{yp},d_1]$ and $[d_1,d',d,z,b,c',P_{yp},d_1]$ (case 3).
- $ud \notin E$; otherwise the chordless cycles $[u,d,c,c',P_{yx},u]$ and $[u,d,z,b,c',P_{yx},u]$ are of different parities.
- $ud' \notin E$ (case 3); otherwise the chordless cycles $[u,d',d,c,c',P_{yx},u]$ and $[u,d',d,z,b,c',P_{yx},u]$ are of different parities.

But then (P8) is contradicted by the pair of paths $[d,c,c',P_{yx},u]$ and $[d,z,b,c',P_{yx},u]$ (case 1), respectively $[d_1,d,c,c',P_{yx},u]$ and $[d_1,d,z,b,c',P_{yx},u]$ (case 2), respectively $[d_1,d',d,c,c',P_{yx},u]$ and $[d_1,d',d,z,b,c',P_{yx},u]$ (case 3).

We show that $N_T(X) \subseteq N(b)$.

We only have to consider $T_{i-1} \cup T_i \cup T_{i+1}$, since by (P4) all the other sets T_j are included in $N(b)$. Assume there exists $q \in T_{i-1} \cup T_i \cup T_{i+1}$ and $x \in X$ such that $qx \in E$,

$qb \notin E$.

- $qc \notin E$; otherwise $qa \notin E$ (else $[q, a, b, c, q]$ is a C_4), and in situation I we have that the chordless cycles $[P_{ax}, q, c, b, a]$, $[P_{ax}, q, z, a]$ are of different cardinalities (once again we assume that x is the neighbour of q closest to a), while in situation II we have that Remark 5 is contradicted (since $qb \notin E$, $qc \in E$ and still $qx \in E$).
- $q \notin T_{i-1}$; otherwise, as $c \in T_{i+1}$, by (P4) we would have $qc \in E$.
- $qc_1 \notin E$; otherwise $[q, z, b, c_1, q]$ is a C_4.
- $qa_1 \notin E$; otherwise $[q, z, b, a_1, q]$ is a C_4.
- $qc' \notin E$; otherwise $[q, z, b, c', q]$ is a C_4.
- $qa \notin E$; otherwise the P_4 $[q, a, b, c]$ is in case (i) or (v) (see the beginning of the proof of Lemma 5). This is not possible.

If $q \in T_i$, then let $Q_1 = N_{U_{i-1}}(q)$. If $q \in T_{i+1}$, let $q' \in U_i$ such that $q'q \in E$ and $Q_1 = N_{U_{i-1}}(q')$. In both cases, denote q_1 an arbitrary vertex of Q_1.

- $q'b \notin E$ (case $q \in T_{i+1}$); otherwise $[q', q, z, b, q']$ is a C_4.
- $q'c \notin E$ (case $q \in T_{i+1}$); otherwise $[q', q, z, c, q']$ is a C_4.
- $q'c' \notin E$ (case $q \in T_{i+1}$); otherwise $[q', q, z, c, c', q']$ is a C_5.
- $q'c_1 \notin E$ (case $q \in T_{i+1}$); otherwise $[q', q, z, b, c_1, q']$ is a C_5.
- $q_1 b \notin E$; otherwise $[q_1, q, z, b, q_1]$ is a C_4 (case $q \in T_i$), respectively $[q_1, q', q, z, b, q_1]$ is a C_5 (case $q \in T_{i+1}$).
- $q_1 c \notin E$, since $q_1 \in U_{i-1}$ and $c \in T_{i+1}$.
- $q_1 c' \notin E$; otherwise $q_1 \in C_1$, and since $C_1 \subseteq B_1$ we have a contradiction to $q_1 b \notin E$.

But then (P8) is contradicted by the pair of paths $[q_1, q, z, b, c_1]$ and $[q_1, q, z, c, c', c_1]$ (case $q \in T_i$), respectively $[q_1, q', q, z, b, c_1]$ and $[q_1, q', q, z, c, c', c_1]$ (case $q \in T_{i+1}$).

Consequently, $N(X) \subseteq S$ and therefore X and d are in different connected components of $G - S$. Thus S is a star-cutset, so G is breakable. $\square$

Proof of Claim 4. Again, the proof is almost the same for the two cases, so we treat them in parallel (calling them case 1 when $d \in T_{i+1}$ and case 2 when $d \in T_{i+2}$). Assume G is not breakable.

Because of $H_{i+1} = N_{H_i'}(U_i)$, there exist $a', b' \in U_i$ such that $a'a, b'b \in E$ (cases 1, 2), and there exists $d' \in U_i$ such that $dd' \in E$ (case 1). Moreover, in case 2 there exist (for similar reasons) $d'' \in U_{i+1}$ such that $d''d \in E$ and $d' \in U_i$ such that $d'd'' \in E$.

Define (see Fig. 5).

$A_1 = N_{U_{i-1}}(a')$,
$B_1 = N_{U_{i-1}}(b')$,
$C_1 = N_{U_{i-1}}(c)$ and
$D_1 = N_{U_{i-1}}(d')$.

As usual, these sets are non-empty. Then we have:

- $ca' \notin E$; otherwise $[a, a', c, z, a]$ is a C_4.
- $A_1 \cap C_1 = \emptyset$; otherwise $[c, (ac)_1, a', a, z, c]$ is a C_5.
- $ba' \notin E$; otherwise (P8) is contradicted by $[a_1, a', b, c, c_1]$ and $[a_1, a', a, z, c, c_1]$
- $b' \neq a'$ since $ba' \notin E$.

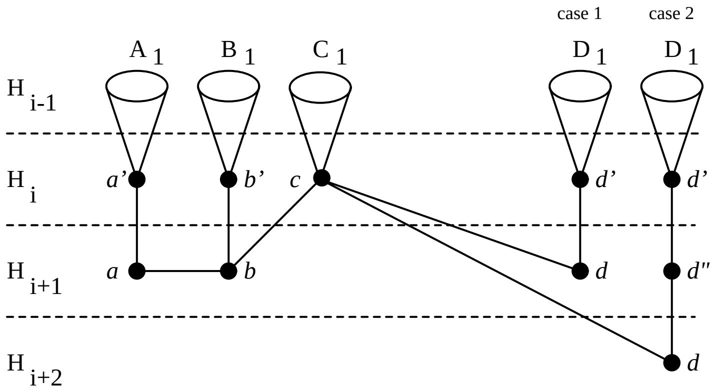

Fig. 5. The sets A_1, B_1, C_1, D_1 in the two cases with respect to d.

- $b'a \notin E$ since in the contrary case we successively have $b'c \notin E$ (otherwise $[b', a, z, c, b']$ is a C_4), $B_1 \cap C_1 = \emptyset$ (otherwise $[c, (bc)_1, b', b, c]$ is a C_4) and then (P8) is contradicted by $[b_1, b', b, c, c_1]$ and $[b_1, b', a, z, c, c_1]$.
- $b'a' \notin E$; otherwise $[b', b, a, a', b']$ is a C_4.
- $A_1 \cap B_1 = \emptyset$; otherwise $[(ab)_1, a', a, b, b', (ab)_1]$ is a C_5.
- $a'd \notin E$; otherwise $[a, a', d, z, a]$ is a C_4.
- $ad'' \notin E$ (case 2); otherwise $[a, d'', d, z, a]$ is a C_4.
- $ad' \notin E$; otherwise $[a, d', d, z, a]$ is a C_4 (case 1), respectively $[a, d', d'', d, z, a]$ is a C_5 (case 2).
- $a'd'' \notin E$ (case 2); otherwise $[a, a', d'', d, z, a]$ is a C_5.
- $a' \neq d'$ since $a'd \notin E$ (case 1), respectively $a'd'' \notin E$ (case 2).
- $bd'' \notin E$ (case 2); otherwise $[b, d'', d, z, b]$ is a C_4.
- $bd' \notin E$; otherwise $[b, d', d, z, b]$ is a C_4 (case 1), respectively $[b, d', d'', d, z, b]$ is a C_5 (case 2).
- $b' \neq d'$ since $bd' \notin E$.
- $b'd \notin E$; otherwise $[b', d, z, b, b']$ is a C_4.
- $b'd'' \notin E$ (case 2); otherwise $[b, b', d'', d, z, b]$ is a C_5.
- $a'd' \notin E$; otherwise $[a, a', d', d, z, a]$ is a C_5 (case 1), while in case 2 we successively have $cd'' \in E$ and $cd' \notin E$ (the cycle on seven vertices $[a', a, b, c, d, d'', d', a']$ must have chords, and the only two possible chords are cd'', cd'; but $cd' \in E$ implies that $[a', a, b, c, d', a']$ is a C_5), $C_1 \cap D_1 = \emptyset$ (else $[(cd)_1, d', d'', c, (cd)_1]$ is a C_4), $b'd' \notin E$ (otherwise the cycle $[b', d', d'', c, b, b']$ is a C_5 or contains a C_4), $B_1 \cap D_1 = \emptyset$ (otherwise $[(bd)_1, b', b, z, d, d'', d', (bd)_1]$ is a C_7), $d_1 a' \notin E$ (else (P8) is contradicted by $[d_1, a', a, b, c, c_1]$ and $[d_1, d', d'', c, c_1]$), and then (P8) is contradicted by the pair of paths $[b_1, b', b, a, a', d', d_1]$ and $[b_1, b', b, z, d, d'', d', d_1]$.
- $b'd' \notin E$; otherwise $[b', d', d, z, b, b']$ is a C_5 (case 1), respectively in case 2 we have $b'd_1 \notin E$ (else, as (P8) cannot be contradicted by the paths $[d_1, b', b, a, a', a_1]$ and $[d_1, d', d'', d, c, b, a, a', a_1]$, we have $cd'' \in E$, $cd' \notin E$; consequently $[b', d', d'', c, b, b']$ is a C_5 or contains a C_4) and then (P8) is contradicted by $[a_1, a', a, b, b', d', d_1]$ and $[a_1, a', a, z, d, d'', d', d_1]$.

- $d'c \in E$, since in the contrary case we successively have $C_1 \cap D_1 = \emptyset$ (otherwise $[c,(cd)_1,d',d,c]$ is a C_4 in case 1, respectively $[c,(cd)_1,d',d'',d,c]$ is a C_5 or contains a C_4 in case 2), and property (P8) is contradicted by the pair of paths $[d_1,d',d,c,b,a,a',a_1]$, $[d_1,d',d,z,a,a',a_1]$ (case 1), respectively $[d_1,d',d'',d,c,b,a,a',a_1]$, $[d_1,d',d'',d,z,a,a',a_1]$ in case 2, provided that $d''c \notin E$. If $d''c \in E$ (and we still are in case 2), then $cb_1 \notin E$ (otherwise (P8) is again contradicted by $[b_1,c,d'',d',d_1]$ and $[b_1,b',b,z,d,d'',d',d_1]$), $b'c \in E$ (otherwise (P8) is contradicted by $[b_1,b',b,c,d'',d',d_1]$ and $[b_1,b',b,z,d,d'',d',d_1]$), and (P8) is contradicted by $[a_1,a',a,b,b',b_1]$ and $[a_1,a',a,z,c,b',b_1]$.
- $d''c \in E$ (case 2); otherwise $[d',c,d,d'',d']$ is a C_4.
- $b'c \in E$; otherwise we have $B_1 \cap C_1 = \emptyset$ (else $[b',(bc)_1,c,b,b']$ is a C_4), so in case 1 we have $C_1 \cap D_1 = \emptyset$ (else $[(cd)_1,c,b,a,a',a_1]$ and $[(cd)_1,d',d,z,a,a',a_1]$ contradict (P8)) and (P8) is contradicted by $[b_1,b',b,z,d,d',d_1]$, $[b_1,b',b,c,d',d_1]$, while in case 2 we have $D_1 \subseteq C_1$ (else (P8) is contradicted by $[a_1,a',a,b,c,d',(d.c)_1]$ and $[a_1,a',a,z,d,d'',d',(d.c)_1]$), $b'd_1 \notin E$ (else $B_1 \cap D_1 \neq \emptyset$ and this contradicts $B_1 \cap C_1 = \emptyset$) and (P8) is contradicted by $[b_1,b',b,c,d_1]$, $[b_1,b',b,z,d,d'',d',d_1]$.
- $B_1 \subseteq C_1$; otherwise (P8) is contradicted by $[a_1,a',a,b,b',(b.c)_1]$ and $[a_1,a',a,z,c,b',(b.c)_1]$.
- $N_{T_i}(d) \subseteq \{c\} \cup N(c)$; otherwise, with some $t \in N_{T_i}(d) - \{c\} - N(c)$ and some $t_1 \in N_{U_{i-1}}(t)$, we successively have $tb \notin E$ (else $[t,b,c,d,t]$ is a C_4), $td' \notin E$ (else $[t,d',c,z,t]$ is a C_4), $td'' \notin E$ (case 2, else $[t,d'',c,z,t]$ is a C_4), $d \in T_{i+2}$ (else we are in case 1 and $[b,c,d,t]$ is a P_4 of type iii), $D_1 \subseteq C_1$ (else (P8) is contradicted by the paths $[(d.c)_1,d',d'',d,z,a,a',a_1]$ and $[(d.c)_1,d',c,b,a,a',a_1]$), and then (P8) is contradicted by $[d_1,d',d'',d,t,t_1]$ and $[d_1,c,z,t,t_1]$.
- $N_{T_{i+1}}(d) \subseteq N(c)$; otherwise, with some $t \in N_{T_{i+1}}(d) - N(c)$, some $t' \in N_{U_i}(t)$ and some $t_1 \in N_{U_{i-1}}(t')$, we successively have $tb \notin E$ (else $[t,b,c,d,t]$ is a C_4), $td' \notin E$ (else $[t,d',c,z,t]$ is a C_4), $td'' \notin E$ (case 2, else $[t,d'',c,z,t]$ is a C_4), $t'c \notin E$ (else $[t',c,z,t,t']$ is a C_4), $t'd'' \notin E$ (case 2, else $[t',t,z,c,d'',t']$ is a C_5), $t'd \notin E$ (obviously in case 2, and in case 1, since otherwise t' plays the same role as d' with respect to $[a,b,c,d]$ so $d'c \in E$ implies $t'c \in E$, a contradiction), $t'd' \notin E$ (else $[t',d',c,z,t,t']$ is a C_5), $D_1 \subseteq C_1$ (else, in case 1, $t_1d' \notin E$ because of the cycles $[t_1,t',t,z,c,d',t_1]$ and $[t_1,t',t,d,d',t_1]$ and (P8) is contradicted by the paths $[t_1,t',t,z,c,d',(d.c)_1]$ and $[t_1,t',t,d,d',(d.c)_1]$, while in case 2 (P8) is contradicted by $[(d.c)_1,d',d'',d,z,a,a',a_1]$ and $[(d.c)_1,d',c,b,a,a',a_1]$), $A_1 \cap D_1 = \emptyset$ (else $A_1 \cap C_1 \neq \emptyset$) and then (P8) is contradicted by $[d_1,d',d,z,a,a',a_1]$ and $[d_1,c,b,a,a',a_1]$ (case 1), respectively by $[d_1,d',d'',d,t,t',t_1]$ and $[d_1,c,z,t,t',t_1]$ (case 2).
- $N_T(d) \subseteq \{c\} \cup N(c)$; otherwise there exists $t \in N_T(d) - \{c\} - N(c)$, for which $tb \notin E$ holds (else $[t,b,c,d,t]$ is a C_4), so that $t \in N_{T_i \cup T_{i+1}}(d)$ (otherwise t is adjacent either to b or to c by (P4)). But this is impossible by the two affirmations above.
- $d \in T_{i+1}$; otherwise we are in case 2 and $N(d) \subseteq U_{i+1} \cup T \cup \{z\} \cup V_{i+2}$, while $N_T(d) \subseteq \{c\} \cup N(c)$ (as before), $\{z\} \subseteq N(c)$ (obviously), $N_{U_{i+1}}(d) \subseteq N(c)$ (d' is an *arbitrary* vertex of $N_{U_{i+1}}(d)$) and $V_{i+2} \subseteq N(c)$ (by (P4)). So $N(d) \subseteq N(c)$ and G is breakable.

So we are in case 1. We will prove that $S = \{c\} \cup N(c) - \{d\}$ is a star-cutset, by showing that the connected component of d in the graph induced by $U_{i+1} \cup V_{i+1} - S$, denoted X, is separated in $G-S$ from the rest of the graph. We have:

- $N_{U_i}(d)\subseteq N(c)$, since d' is arbitrarily chosen in $N_{U_i}(d)$.

We show that $N_{U_i}(X)\subseteq N(c)$.

Suppose this is not the case, and let $x\in X$ be a vertex such that $N_{U_i}(x)-N(c)\neq\emptyset$. Then there exists a chordless path P_{dx} joining d to x in the subgraph induced by X in G. Without loss of generality we assume that every vertex p on $P_{dx}-\{x\}$ has $N_{U_i}(p)\subseteq N(c)$ (otherwise we consider the counterexample which is closest to d and we call it x).

Let $q\in N_{U_i}(x)-N(c)$, define
$Q_1=N_{U_{i-1}}(q)$,
and preserve for Q_1 the convention of notation we made for A_1,B_1,C_1,D_1.

We have:

- $qp\notin E$, for every $p\in V(P_{dx})$; otherwise, as $N_{U_i}(p)\subseteq N(c)$, we should have $qc\in E$ too.
- $pb,pb'\notin E$, for every $p\in V(P_{dx})$; otherwise, let p be the counterexample closest to d (which is necessarily different from d) and let p' be the neighbour of p on P_{dp} ($p'=d$ is possible). Then $p'b,p'b'\notin E$ (by the choice of p) and with some $y\in N_{U_i}(p')$ we successively have $yc\in E$ (by the choice of x, since $p'\neq x$), $yb\notin E$ (else take a chordless path P'_{yd} joining y to d in the graph induced by $\{y\}\cup V(P_{p'd})$ and notice that the chordless cycles $[c,P'_{yd},c]$ and $[b,P'_{yd},z,b]$ are of different parities; a contradiction is immediately obtained if P'_{yd} has at least three vertices; if it has only two vertices, $[b,P'_{yd},z,b]$ is a C_4), $bp\notin E$ (else $[b,p,p',y,c,b]$ is a C_5 or contains a C_4), so $b'p\in E$ and the chordless cycles $[b',c,P_{dp},b'],[b',b,z,P_{dp},b']$ have different parities, a contradiction.
- $pa\notin E$, for every $p\in V(P_{dx})\cup\{q\}$; otherwise, if p is the counterexample which is closest to d, the chordless cycles $[a,b,c,P_{dp},a]$, $[a,z,P_{dp},a]$ are of different parities (if $p=q$, then P_{dp} has to be replaced by $P_{dx}q$).
- $pa'\notin E$, for every $p\in V(P_{dx})\cup\{q\}$; otherwise, if p is the counterexample which is closest to d, the chordless cycles $[a',a,b,c,P_{dp},a']$, $[a',a,z,P_{dp},a']$ are of different parities (if $p=q$, then P_{dp} has to be replaced by $P_{dx}q$).
- $pa_1\notin E$, for every $p\in V(P_{dx})\cup\{q\}$; this is obviously true for $p\in V(P_{dx})$ since $a_1\in U_{i-1}$ while $p\in H_{i+1}$; if $p=q$ then the chordless cycles $[a_1,a',a,b,c,P_{dx},q,a_1]$, $[a_1,a',a,z,P_{dx},q,a_1]$ are of different parities.
- $q_1c\notin E$; otherwise $[q_1,c,P_{dx},q,q_1]$ implies that P_{dx} is even, so that (P8) is contradicted by $[q_1,c,b,a,a',a_1]$ and $[q_1,q,P_{xd},z,a,a',a_1]$.

But then (P8) is contradicted by $[q_1,q,P_{xd},z,a,a',a_1]$ and $[q_1,q,P_{xd},c,b,a,a',a_1]$.

We show that $N_T(X)\subseteq N(c)$.

If this is not the case, take $x\in X$ such that $N_T(x)-N(c)\neq\emptyset$ and let $t\in N_T(x)-N(c)$. Then $t\neq d$. With $y\in N_{U_i}(x)$, we have $yc\in E$ and then $[y,c,z,t,x,y]$ is a C_5 or contains a C_4.

Now, since $N(X)\subseteq N_{U_i}(X)\cup T\cup N_{H_{i+2}}(X)\cup\{z\}$ and all the sets in the right part of the expression are included in $\{c\}\cup N(c)$, we have $N(X)\subseteq\{c\}\cup N(c)$. Then $S=\{c\}\cup N(c)-\{d\}$ is a star-cutset. □

Proof of Claim 5. By contradiction, assume the claim does not hold, and let $[a,b,c,d]$ be a P_4 as indicated in the hypothesis.

Because of $H_{p-1}=N_{H'_{p-2}}(U_{p-2})$, there exists $a'\in U_{p-2}$ such that $a'a\in E$. Moreover, for similar reasons there exist $b''\in U_{p-1}, b'\in U_{p-2}$ such that $bb'', b''b'\in E$, and there exist $d''\in U_{p-1}$, $d'\in U_{p-2}$ such that $dd'', d''d'\in E$.

Define

$A_1=N_{U_{p-3}}(a')$,
$B_1=N_{U_{p-3}}(b')$,
$C_1=N_{U_{p-3}}(c)$ and
$D_1=N_{U_{p-3}}(d')$.

As usual, these sets are non-empty. Then we have:

- $ca'\notin E$; otherwise $[a,a',c,z,a]$ is a C_4.
- $da'\notin E$; otherwise $[a,a',d,z,a]$ is a C_4.
- $A_1\cap C_1=\emptyset$; otherwise $[c,(ac)_1,a',a,z,c]$ is a C_5.
- $ad''\notin E$; otherwise $[a,d'',d,z,a]$ is a C_4.
- $ad'\notin E$; otherwise $[a,d',d'',d,z,a]$ is a C_5.
- $a'd''\notin E$; otherwise $[a,a',d'',d,z,a]$ is a C_5.
- $a'\neq d'$ since $a'd''\notin E$.
- $bd''\notin E$; otherwise $[b,d'',d,z,b]$ is a C_4.
- $bd'\notin E$; otherwise $[b,d',d'',d,z,b]$ is a C_5.
- $b''d\notin E$; otherwise $[b,b'',d,z,b]$ is a C_4.
- $b'd\notin E$; otherwise $[b,b'',b',d,z,b]$ is a C_5.
- $b''\neq d''$ since $b''d\notin E$;
- $b''d''\notin E$; otherwise $[b'',d'',d,z,b,b'']$ is a C_5.
- $b'a\notin E$ since in the contrary case we successively have $b''a\in E$ (else $[b',a,b,b'',b']$ is a C_4), $b''c\notin E$ (otherwise $[b'',a,z,c,b'']$ is a C_4), $b'c\notin E$ (otherwise $[b',c,b,b'',b']$ is a C_4), $B_1\cap C_1=\emptyset$ (otherwise $[c,(bc)_1,b',b'',b,c]$ is a C_5), $b'd''\notin E$ (otherwise $[b',d'',d,z,a,b']$ is a C_5), $b'd'\notin E$ (otherwise the cycle $[b',d',d'',d,c,b,a,b']$ which must have chords implies that $cd'\notin E$, $cd''\in E$, and the cycle $[b',d',d'',d,z,b,b'',b']$ which must have chords implies $b''d'\in E$; then (P8) is contradicted by $[d_1,d',d'',c,c_1]$ and $[d_1,d',b'',b,c,c_1]$), $b''d'\notin E$ (else $[b_1,b',b'',d',d_1]$ and $[b_1,b',a,z,d,d'',d',d_1]$ contradict (P8)). But then (P8) is contradicted by $[b_1,b',b'',b,z,d,d'',d',d_1]$ and $[b_1,b',a,z,d,d'',d',d_1]$.
- $b'\neq a'$ since $b'a\notin E$.
- $b''a\notin E$ since in the contrary case we successively have $b''c\notin E$ (otherwise $[b'',a,z,c,b'']$ is a C_4), $b'c\notin E$ (otherwise $[b',c,b,b'',b']$ is a C_4), $B_1\cap C_1=\emptyset$ (otherwise $[c,(bc)_1,b',b'',b,c]$ is a C_5) and then (P8) is contradicted by $[b_1,b',b'',b,c,c_1]$ and $[b_1,b',b'',a,z,c,c_1]$.
- $b''a'\notin E$; otherwise $[a',b'',b,a,a']$ is a C_4.
- $b'a'\notin E$; otherwise $[b',b'',b,a,a',b']$ is a C_5.
- $a'd'\notin E$; otherwise we successively have $cd''\in E$ and $cd'\notin E$ (the cycle on seven vertices $[a',a,b,c,d,d'',d',a']$ must have chords, and the only two possible chords are cd'',cd'; but $cd'\in E$ implies that $[a',a,b,c,d',a']$ is a C_5), $b''d'\notin E$ (otherwise

the cycle $[b'', d', d'', c, b, b'']$ is a C_5 or contains a C_4), $b'd'' \notin E$ (otherwise (P8) is contradicted by $[b_1, b', d'', d, z, a, a', a_1]$ and $[b_1, b', b'', b, a, a', a_1]$); $b'd' \notin E$ (otherwise $[b, b'', b', d', d'', d, z, b]$ is a C_7); $D_1 \cap A_1 = \emptyset$ (otherwise (P8) is contradicted by $[c_1, c, d'', d', (da)_1]$ and $[c_1, c, b, a, a', (da)_1]$); $B_1 \cap A_1 = \emptyset$ (otherwise (P8) is contradicted by $[d_1, d', a', (ab)_1]$ and $[d_1, d', d'', d, z, b, b'', b', (ab)_1]$), and then (P8) is contradicted by the pair of paths $[b_1, b', b'', b, a, a', d', d_1]$ and $[b_1, b', b'', b, z, d, d'', d', d_1]$.

- $b''d' \notin E$; otherwise (P8) is contradicted by $[a_1, a', a, b, b'', d', d_1]$ and $[a_1, a', a, z, d, d'', d', d_1]$.
- $b' \neq d'$ since $b''d' \notin E$.
- $b'd'' \notin E$; otherwise (P8) is contradicted by $[a_1, a', a, b, b'', b', b_1]$ and $[a_1, a', a, z, d, d'', b', b_1]$.
- $b'd' \notin E$; otherwise $[b', d', d'', d, z, b, b'', b']$ is a C_7.
- $d''c \in E$, since in the contrary case $d'c \notin E$ (otherwise $[c, d', d'', d, c]$ is a C_4), $D_1 \cap C_1 = \emptyset$ (otherwise $[c, (cd)_1, d', d'', d, c]$ is a C_5) and (P8) is contradicted by $[d_1, d', d'', d, c, b, a, a', a_1]$ and $[d_1, d', d'', d, z, a, a', a_1]$.
- $d'c \notin E$; in the contrary case, (P8) is contradicted by $[d_1, d', c, b, a, a', a_1]$ and $[d_1, d', d'', d, z, a, a', a_1]$, except if $D_1 \subseteq C_1$. In this last case we successively have $cb'' \in E$ and $cb' \notin E$ (otherwise (P8) is contradicted by $[d_1, c, b, b'', b', b_1]$ and $[d_1, d', d'', d, z, b, b'', b', b_1]$, and $cb' \in E$ does not change this), so (P8) is contradicted by $[a_1, a', a, b, b'', b', b_1]$ and $[a_1, a', a, z, c, b'', b', b_1]$.

But then we have $cd_1 \notin E$ (otherwise $[c, d_1, d', d'', c]$ is a C_4), $cb'' \in E$ and $cb' \notin E$ (otherwise (P8) is contradicted by $[d_1, d', d'', c, b, b'', b', b_1]$ and $[d_1, d', d'', d, z, b, b'', b', b_1]$, and $cb' \in E$ does not change this), $cb_1 \notin E$ (otherwise $[c, b'', b', b_1, c]$ is a C_4), so (P8) is contradicted by $[a_1, a', a, b, b'', b', b_1]$ and $[a_1, a', a, z, c, b'', b', b_1]$. □

Proof of Claim 6. Suppose the contrary holds, and say that we are in case 1 if $d \in T_{p-2}$ and in case 2 if $d \in T_{p-1}$.

Because of $H_{p-1} = N_{H'_{p-2}}(U_{p-2})$, there exists $a' \in U_{p-2}$ such that $a'a \in E$ (cases 1, 2), and there exists $d' \in U_{p-2}$ such that $dd' \in E$ (case 2).

Define (see Fig. 6)

$A_1 = N_{U_{p-3}}(a')$.
$B_1 = N_{U_{p-3}}(b)$ and
$D_1 = N_{U_{p-3}}(d)$ (case 1), respectively $D_1 = N_{U_{p-3}}(d')$ (case 2),

As usual, these sets are non-empty. Then we have:

- $d'b \notin E$ (case 2); otherwise $[d', b, z, d, d']$ is a C_4.
- $d'a \notin E$ (case 2); otherwise $[d', a, z, d, d']$ is a C_4.
- $a'd \notin E$; otherwise $[d, a', a, z, d]$ is a C_4.
- $a'd' \notin E$ (case 2); $[d, d', a', a, z, d]$ is a C_5.
- $D_1 \cap B_1 = \emptyset$; otherwise $[d, (db)_1, b, z, d]$ is a C_4 (case 1), respectively $[d, d', (db)_1, b, z, d]$ is a C_5 (case 2).

Let now X be the connected component of a in the subgraph induced by $U_{p-1} \cup V_{p-1} - S$ (recall that $S = \{b\} \cup N(b) - \{a\}$). We will show that, in $G - S$, X is

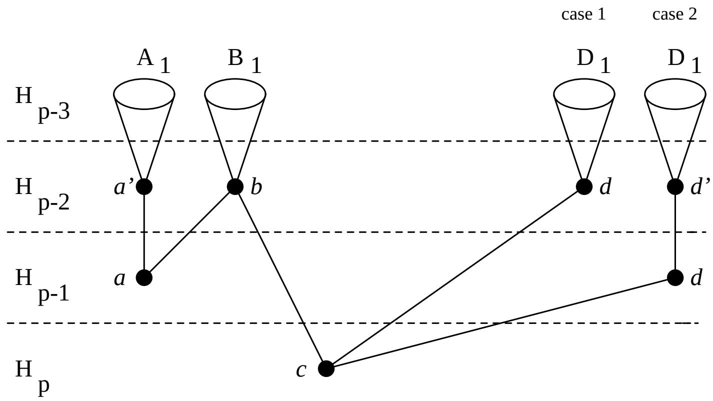

Fig. 6. The sets A_1, B_1, D_1 in the two cases with respect to d.

disconnected from the rest of the graph. Recall that, by the definition of U_{p-1}, V_{p-1}, no vertex in X is adjacent to z.

- $a'b \in E$; in the contrary case we have $a_1 b \notin E$ (otherwise $[b, a_1, a', a, b]$ is a C_4) and (P8) is contradicted by $[a_1, a', a, b, c, d, d_1]$ and $[a_1, a', a, z, d, d_1]$ (case 1), respectively $[a_1, a', a, b, c, d, d', d_1]$ and $[a_1, a', a, z, d, d', d_1]$ (case 2).
- $N_{U_{p-2}}(a) \subseteq N(b)$, since a' was arbitrarily chosen in $N_{U_{p-2}}(a)$, and $a'b \in E$.
- $N_T(a) \subseteq N(b)$; otherwise for a counterexample t we successively have $tc \notin E$ (otherwise $[a, t, c, b, a]$ is a C_4), $t \notin T_{p-2}$ (since $tc \notin E$, by (P4)), $td \notin E$ (otherwise $[a, t, d, c, b, a]$ is a C_5), $t \notin T_p$ (since $tb \notin E$, by (P4)), $t \in T_{p-1}$ (since only T_{p-2}, T_{p-1}, T_p are non-empty, and $t \notin T_{p-2} \cup T_p$ as above), so $[t, a, b, c]$ is a P_4 of type (x), a contradiction (such a P_4 cannot appear).

We show that $N_{U_{p-2}}(X) \subseteq N(b)$.

Suppose this is not the case, and let $x \in X$ be a vertex such that $N_{U_{p-2}}(x) - N(b) \neq \emptyset$. Then there exists a chordless path P_{ax} joining a to x in the subgraph induced by X in G. Without loss of generality, we assume that every vertex p on $P_{ax} - \{x\}$ has $N_{U_{p-2}}(p) \subseteq N(b)$ (otherwise, we consider the counterexample which is closest to a and we call it x).

Let $q \in N_{U_{p-2}}(x) - N(b)$, define
$Q_1 = N_{U_{p-3}}(q)$,
and preserve for Q_1 the convention of notation we made for A_1, B_1, C_1, D_1.

We have:

- $qa \notin E$; otherwise, as $N_{U_{p-2}}(a) \subseteq N(b)$, we should have $qb \in E$ too.
- $pc \notin E$, for every $p \in V(P_{ax})$; otherwise, let p be a counterexample (which is necessarily different from a) and consider two cases. If $p \neq x$, then let $p' \in N_{U_{p-2}}(p)$, and notice that $p'b \in E$ (by the choice of x), so $[b, p', p, c, b]$ is a C_4. If $p = x$, let x' be the vertex preceding x on P_{ax} (possibly, $x' = a$), let $x'' \in N_{U_{p-2}}(x')$ and notice that $x''b \in E$ (by the choice of x), $x''x \notin E$ (otherwise $[x, x'', b, c, x]$ is a C_4), so $[b, x'', x', x, c, b]$ is a C_5.

- $pd \notin E$, for every $p \in V(P_{ax})$; otherwise, if p is the counterexample which is closest to a, the chordless cycles $[p,d,c,b,P_{ap}]$, $[p,d,z,P_{ap}]$ are of different parities.
- $pd' \notin E$, for every $p \in V(P_{ax})$ (case 2); otherwise, if p is the counterexample which is closest to a, the chordless cycles $[p,d',d,c,b,P_{ap}]$, $[p,d',d,z,P_{ap}]$ are of different parities.
- $qd \notin E$; otherwise the chordless cycles $[q,d,c,b,P_{ax},q]$, $[q,d,z,P_{ax},q]$ are of different parities.
- $qd' \notin E$ (case 2); otherwise, the chordless cycles $[q,d',d,c,b,P_{ax},q]$, $[q,d',d,z,P_{ax},q]$ are of different parities.
- $q_1b \in E$; otherwise (P8) is contradicted by $[q_1,q,P_{xa},b,c,d,d_1]$ and $[q_1,q,P_{xa},z,d,d_1]$ (case 1), respectively $[q_1,q,P_{xa},b,c,d,d',d_1]$ and $[q_1,q,P_{xa},z,d,d',d_1]$ (case 2).
- P_{ax} is even, since the chordless cycle $[a,b,q_1,q,P_{xa}]$ must be even.

But then (P8) is contradicted by $[q_1,q,P_{xa},z,d,d_1]$ and $[q_1,b,c,d,d_1]$ (case 1), respectively by $[q_1,q,P_{xa},z,d,d',d_1]$ and $[q_1,b,c,d,d',d_1]$ (case 2).

We show that $N_T(X) \subseteq N(b)$.
Let $x \in X$ such that $N_T(x) - N(b) \neq \emptyset$, and let $s \in N_T(x) - N(b)$. We have:

- $s \notin T_p$, since by (P4) b is adjacent to all T_p.
- $sa \notin E$, since $N_T(a) \subseteq N(b)$

But then with $v \in N_{U_{p-2}}(x)$ we have $vb \in E$ (since $N_{U_{p-2}}(X) \subseteq N(b)$), so that $[x,s,z,b,v,x]$ is a C_5 (if $sv \notin E$) or contains a C_4 (if $sv \in E$).
Now, recall that $N(X) \subseteq U_{p-2} \cup U_{p-1} \cup V_{p-1} \cup V_p \cup T \cup \{z\}$. We have proved that $N_{U_{p-2}}(X) \cup N_T(X) \subseteq N(b)$ and it is obvious that $V_p \cup \{z\} \subseteq N(b)$ (by (P4)). To complete the proof that $N(X) \subseteq N(b)$, we only have to notice that every neighbour of X in U_{p-1} or V_{p-1} has to be in $N(b)$, otherwise it should have been put in X (by the definition of X). $\square$

5. Corollaries and open questions

From Theorem 1, it is now easy to deduce that:

Corollary 1. *Every unbreakable C_4-free Berge graph (with two vertices or more) which is not a clique contains at least two non-adjacent loose vertices.*

Proof of Corollary 1. Perform LexBFS on G starting with a non-universal vertex w. Then $z = \sigma(1)$ is a loose vertex, which is not universal (since $wz \notin E$). Perform again LexBFS on G, starting with z, and call z' the last vertex chosen by the algorithm. Then z, z' are two non-adjacent loose vertices in G. $\square$

Corollary 2. *Every C_4-free Berge graph has an order $[v_1, v_2, \ldots, v_n]$ of vertices such that, for each $i = 1, 2, \ldots$, the graph induced by $[v_i, v_{i+1}, \ldots, v_n]$ either has v_i as a loose vertex, or is breakable.*

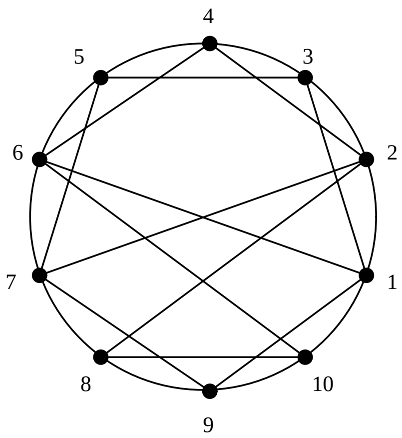

Fig. 7.

Proof of Corollary 2. The order $[\sigma(1), \sigma(2), \ldots, \sigma(n)]$ has the indicated properties, since Theorem 1 can be applied to every induced subgraph $[v_i, v_{i+1}, \ldots, v_n]$. □

Minimal imperfect graphs G have the properties required in the hypothesis of Corollary 1, so they have two non-adjacent loose vertices. However, deducing from this information the perfection of C_4-free Berge graphs is an open problem. If we consider $N(z)$, we have (see [13]) that the vertices in $N(z)$ induce a non-connected graph either in G or in $\bar{G}$. The last one cannot hold, since then G would be breakable. Thus $N(z)$ induces in G a non-connected graph of a particular form (since every connected component is P_4-free), but not particular enough to imply easy conclusions with respect to the perfectness. In fact, even if the connected components were cliques, the conclusion would be difficult to reach, since the conjecture below [5] is still unsettled:

Conjecture. No minimal imperfect graph (except for C_{2k+1}) has a vertex whose neighbourhood is the union of vertex-disjoint cliques.

It is tempting to think that results as the one we need to conclude on the perfection of Berge C_4-free graphs, or the one suggested by the previous conjecture could be approached using partitionable graphs (see [4] for the definition), a class which strictly contains minimal imperfect graphs. Often, proving perfection is done by showing that no partitionable graph with the desired properties exists (as a consequence, no minimal imperfect graph with these properties exists). In our case, trying to show that no partitionable graph (except for the odd holes) with an order as described in Corollary 2 exists will lead to a failure. As an example, consider the partitionable graph in Fig. 7 (given in [4]), with the order $[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]$.

Therefore, to prove perfection it is necessary either to use again the property that G is (C_4, holes)-free, or to explicitly use properties of minimal imperfect graphs which are not properties of partitionable graphs, or else to use the observation that for a C_4-free

Berge graph *every* application of LexBFS gives an order as the one in Corollary 2 (for the graph in Fig. 7, the order $[\sigma(1)=9,\ \sigma(2)=10,\ \sigma(3)=8,\ \sigma(4)=7,\ \sigma(5)=6,\ \sigma(6)=1,\ \sigma(7)=2,\ \sigma(8)=5,\ \sigma(9)=4,\ \sigma(10)=3]$ is obtained by LexBFS, but $N(9)$ contains a P_4 and G is not breakable).

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